ISSN 2229-5518

A half-space problem in the theory of fractional order thermoelasticity with diffusion.

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Abstract We consider in this work the problem of thermoelastic half-space with a permeating substance in contact with the bounding plane by employing the fractional order theory of thermoelasticity, the bounding surface of the half-space is taken to be traction free and subjected to a time dependent thermal shock. The chemical potential is also assumed to be a known function of time on the bounding plane. for the inversion of the Laplace transform based on Fourier expansion techniques. The temperature, displacement, stress, and concentration as well as the chemical potential are obtained. Numerical computations are carried out and represented graphically.

Index Terms—Generalized thermoelasticity, Thermal shock, Thermoelastic diffusion, fractional order thermoelasticity, diffusion.

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1. Introduction

Biot [1] developed the coupled theory of thermoelasticity to deal with a defect of the uncoupled theory that mechanical causes have no effect on the temperature. However, this theory shares a defect of the uncoupled theory in that it predicts infinite speeds of propagation for heat waves.

Lord and Shulman [2] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. This theory was extended by Dhaliwal and Sherief [3] to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and coupled theories of thermoelasticity. For this theory, Ignaczak [4] studied uniqueness of solution; Sherief [5] proved uniqueness and stability. Anwar and Sherief [6] and Sherief [7] developed the state-space approach to this theory. Anwar and Sherief [8] completed the integral equation formulation. Sherief and Hamza [9, 10] solved some two-dimensional problems and studied wave propagation. Sherief and El-Maghraby [11, 12] solved two crack problems. Sherief [13] solved thermoelastic half-space with a permeating substance in contact with the bounding plane in the context of the theory of

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generalized thermoelastic diffusion with one relaxation time.

El-Maghraby [14–16] solved some two-dimensional problems for media affected by heat sources and body forces.

Diffusion can be defined as the random walk, of an ensemble of particles, from regions of high concentration to regions of lower concentration. There is now a great deal of interest in the

study of this phenomenon, due to its many applications in geophysics and industrial applications. In integrated circuit fabrication, diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substrate. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, and form the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors. In most of these applications, the concentration is calculated using what is known as Fick's law. This is a simple law that does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of the temperature on this interaction.

Nowacki [17–20] developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. Recently, Sherief et al. [21] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves.

Fractional calculus has been used successfully to modify many existing models of physical processes. The first application of fractional derivatives was given by Abel who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. One can state that the whole theory of fractional derivatives and integrals was established in the 2nd half of the 19th century. Caputo and Mainardi [22-25] found good agreement with experimental results when using fractional derivatives for description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity. Right now there are five different generalizations of the coupled theory of thermoelasticity the details can be found in Hetnarski and Ignaczak [26]. All five theories are based on assumptions of one kind or another. Also, all these theories model the problem of heat conductions in solids as a purely wave propagation phenomenon. Povstenko [13] made a review of thermoelasticity that uses fractional heat conduction equation. The theory of

thermal stresses based on the heat conduction equation with the Caputo time-fractional derivative is used by Povstenko [27] to investigate thermal stresses in an infinite body with a circular cylindrical hole. Povstenko proposed and investigated new models that use fractional derivative in [28,30]. Sherief et al. [31] developed a new theory of thermoelasticity is derived using the methodology of fractional calculus, proved a uniqueness theorem and derived a reciprocity relation and a variational principle. The theories of coupled thermoelasticity and of generalized thermoelasticity with one relaxation time follow as limit cases. A uniqueness theorem for this model is proved. A variational principle and a reciprocity theorem are derived.

Sherief and Abd El-Latief [32]applied the fractional order theory of thermoelasticity to a 2D problem for a half-space.

2. Formulation of The Problem

We consider the problem of an isotropic

thermoelastic half-space ($x \geq 0$) with a permeating substance (such as a gas) in contact with the upper plane of the half-space ($x = 0$). The x-axis is taken perpendicular to the upper plane pointing inwards. This upper plane of the half-space is taken to be traction free and is subjected to a time- dependent thermal shock. The chemical potential is also assumed to be a known function of time on the upper plane. All considered functions are assumed to be bounded and vanish as $x \to \infty$.

The equation of motion in the absence of body forces is given by [22]

$$\rho\, \ddot{u}_i = \mu\, u_{i,jj} + (\lambda + \mu)\, u_{j,ij} - \beta_1\, \theta_{,i} - \beta_2\, C_{,i}\ ., \tag{1}$$

where ui are the components of the displacement vector, T is the absolute temperature, C is the concentration of the diffusive material in the elastic body, λ, μ are Lamé's constants, ρ is the density, and β_1 and β_2 are the material constants given by

$$\beta 1 = (3\lambda+2\mu)\,\alpha t \text{ and } \beta 2 = (3\lambda+2\mu)\,\alpha c\ ,$$

α_t is the coefficient of linear thermal expansion, and

α_c is the coefficient of linear diffusion expansion.

The energy equation has the form [21] and it can be written in fraction order as:

$$k\, \mathscr{D}^2 T = \rho c_E \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} T}{\partial t^{1+\alpha}} \right) + \beta_1 T_0 \left(\frac{\partial e}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} e}{\partial t^{1+\alpha}} \right) + a T_0 \left(\frac{\partial C}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} C}{\partial t^{1+\alpha}} \right), \tag{2}$$

where k is the thermal conductivity, $0 \leq \alpha \leq 1$, c_E is the specific heat at constant strain, τ_0 is the thermal relaxation time, 'a' is a measure of the thermodiffusion effect, T_0 is a reference temperature assumed to obey the inequality $\left|(T - T_0)/T_0\right| << 1$ and e_{ij} are the components of the strain tensor given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\ .$$

The diffusion equation has the form Sherief et al (2004)

$$D\beta_2 e_{kk,ii} + D a T_{,ii} + \dot{C} + \tau \ddot{C} - D b C_{,ii} = 0\ ,$$

where D is the diffusion coefficient, b is a measure of diffusive effect and τ is the diffusion relaxation time.

The constitutive equations have the form Sherief et al (2004)

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}\left[\lambda e_{kk} - \beta_1 (T - T_0) - \beta_2 C\right], \tag{5a}$$

$$P = -\beta_1 e_{kk} + b C - a\ (T - T_0)\ , \tag{5b}$$

where σ_{ij} are the components of the stress tensor and P is the chemical potential.

It follows from the description of the problem that all considered functions will depend on x and t only. We thus obtain the displacement components of the form,

$$u_x = u(x,t)\ ,\ u_y = u_z = 0\ .$$

The strain components are given by

$$e_{xx} = \mathscr{D}u\ ,\ e_{yy} = e_{zz} = e_{xy} = e_{yz} = e_{zx} = 0$$

where $\mathscr{D} = \dfrac{\partial}{\partial x}$

The cubical dilatation $e = e_{kk}$ is equal to

$$e = \mathscr{D}\, u\ .$$

ISSN 2229-5518

From equation (5a), it follows that the stress tensor components have the form

$$\sigma = \sigma_{xx} = (\lambda + 2\mu)\,\mathscr{D}\,u - \beta_1\,\theta - \beta_2\,C\,,$$

$$\sigma_{yy} = \sigma_{zz} = \lambda\,\mathscr{D}\,u - \beta_1\,\theta - \beta_2\,C\,,$$

$$\sigma_{xy} = \sigma_{zy} = \sigma_{xz} = 0$$

Equations (1), (2) and (4) thus reduce to

$$\rho\,\ddot{u} = \mu\,\mathscr{D}^2 u + (\lambda+\mu)\mathscr{D} e - \beta_1 \mathscr{D} T - \beta_2 \mathscr{D} C \quad ,(10)$$

$$k\,\mathscr{D}^2 T = \rho c_E\left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} T}{\partial t^{1+\alpha}}\right) + \beta_1 T_0\left(\frac{\partial e}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} e}{\partial t^{1+\alpha}}\right) + aT_0\left(\frac{\partial C}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} C}{\partial t^{1+\alpha}}\right), \quad (11)$$

$$D\beta_2\,\mathscr{D}^2 e + D a\,\mathscr{D}^2 T + \dot{C} + \tau\,\ddot{C} - D b\,\mathscr{D}^2 C = 0. \quad (12)$$

By using equation (7), equations (10) - (12) can be written as

$$\rho\,\frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\mathscr{D}\,e - \beta_1\,\mathscr{D}\,T - \beta_2 \mathscr{D}\,C\,, \quad (13)$$

$$k\,\mathscr{D}^2 T = \rho c_E\left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} T}{\partial t^{1+\alpha}}\right) + \beta_1 T_0\left(\frac{\partial e}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} e}{\partial t^{1+\alpha}}\right) + aT_0\left(\frac{\partial C}{\partial t} + \tau_0 \frac{\partial^{1+\alpha} C}{\partial t^{1+\alpha}}\right), \quad (14)$$

$$D\,\beta_2\,\mathscr{D}^2 e + D\,a\,\mathscr{D}^2 T + \left(\frac{\partial}{\partial t} + \tau\,\frac{\partial^2}{\partial t^2} - D\,b\,\mathscr{D}^2\right) C = 0\,. \quad (15)$$

The governing equations can be put in a more convenient form by using the following non-dimensional variables

$$x^* = c_1 \eta\, x\,, \quad u^* = c_1 \eta\, u\,, \quad t^* = c_1^{2\alpha} \eta^{\alpha}\, t\,,$$

$$\tau_0^* = c_1^{2\alpha} \eta^{\alpha}\, \tau_0\,, \quad \tau^* = c_1^{2\alpha} \eta^{\alpha}\, \tau\,,$$

$$\theta^* = \frac{\beta_1 (T - T_0)}{\lambda + 2\mu}\,, \quad C^* = \frac{\beta_2 C}{\lambda + 2\mu}\,, \quad P^* = \frac{P}{\beta_2}\,, \quad \sigma_{ij}^* = \frac{\sigma_{ij}}{\lambda + 2\mu}\,,$$

where $c_1^2 = (\lambda + 2\mu)/\rho\,,\ \eta = \rho c_E / k\,.$ (8a)

(8b)

Using the above non-dimensional variables equations (13) - (15), take the following form where we have dropped the asterisks for convenience (9)

$$\ddot{u} = \mathscr{D}^2 u - \mathscr{D}\,\theta - \mathscr{D}\,C\,, \quad (16)$$

$$\mathscr{D}^2 \theta = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right)\left(\theta + \varepsilon\, e + \varepsilon\,\alpha_1 C\right), \quad (17)$$

$$\mathscr{D}^2 e + \alpha_1 \mathscr{D}^2 \theta + \alpha_2\left(\dot{C} + \tau\,\ddot{C}\right) - \alpha_3 \mathscr{D}^2\,C = 0, \quad (18)$$

where

$$\varepsilon = \frac{\beta_1^2 T_0}{\rho c_E(\lambda + 2\mu)}\,, \quad \alpha_1 = \frac{a(\lambda + 2\mu)}{\beta_1 \beta_2}\,,$$

$$\alpha_2 = \frac{\lambda + 2\mu}{\beta^2 D \eta}\,, \quad \alpha_3 = \frac{b(\lambda + 2\mu)}{\beta_2^2}\,.$$

Also equations (5b) and (8) take the form

$$\sigma_{xx} = e - \theta - C\,,$$

$$\sigma_{yy} = \sigma_{zz} = \left(1 - \frac{2}{\beta^2}\right) e - \theta - C\,,$$

$$P = \alpha_3 C - e - \alpha_1 \theta\,,$$

where $\beta^2 = (\lambda + 2\mu)/\mu$.

The initial conditions of the problem are taken to be homogeneous while the boundary conditions are

$$\sigma(x,t)|_{x=h} = 0\,,\quad u(x,t)|_{x=\infty} = 0 \quad ,(21)$$

$$\theta(x,t)|_{x=0}h = f_1(t)\ ,\ \frac{\partial\theta(x,t)}{\partial x}|_{x=\infty} = 0\ , \tag{22}$$

$$P(x,t)|_{x=h} = f_2(t)\ ,\frac{\partial C(x,t)}{\partial x}|_{x=\infty} = 0\ , \tag{23}$$

where $f_1(t)$ and $f_2(t)$ are known functions of t .

where $f_1(t)$ and $f_2(t)$ are known functions of t . This means that the upper surface is traction free and acted upon by two shocks.

2.1. SOLUTION IN THE LAPLACE TRANSFORM DOMAIN

Introducing the Laplace transform defined by the formula

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt\ ,$$

into equations (16)-(19) and (20) and using the homogeneous initial conditions, we obtain

$$s^2\bar{u} = \mathscr{D}^2\bar{u} - \mathscr{D}\bar{\theta} - \mathscr{D}\bar{C}\ , \tag{24}$$

$$\mathscr{D}^2\bar{\theta} = \left(s + \tau_0 s^{1+\alpha}\right)\left[\bar{\theta} + \varepsilon\bar{e} + \varepsilon\alpha_1\bar{C}\right], \tag{25}$$

$$\mathscr{D}^2\bar{e} + \alpha_1\mathscr{D}^2\bar{\theta} + \left[\alpha_2\left(s + \tau s^2\right) - \alpha_3\mathscr{D}^2\right]\bar{C} = 0\ , \tag{26}$$

$$\bar{\sigma}_{xx} = \bar{e} - \bar{\theta} - \bar{C}\ , \tag{27a}$$

$$\sigma_{yy} = \sigma_{zz} = \left(1 - \frac{2}{\beta^2}\right)e - \theta - \alpha_1 C\ , \tag{27b}$$

$$\bar{P} = \alpha_3\bar{C} - \bar{e} - \alpha_1\bar{\theta} \tag{28}$$

Taking the divergence of equation (24), we obtain

$$(\mathscr{D}^2 - s^2)\bar{e} - \mathscr{D}^2\bar{\theta} - \mathscr{D}^2\bar{C} = 0\ . \tag{29}$$

Eliminating $\bar{e}$ and $\bar{C}$ between equations (25), (26) and (29), we obtain

$$(\mathscr{D}^6 - a_1\mathscr{D}^4 + a_2\mathscr{D}^2 - a_3)\bar{\theta} = 0\ ., \tag{30}$$

where

$$a_1 = \frac{s}{\alpha_3 - 1}\left[(1+\tau_0 s^\alpha)\left(\alpha_1\varepsilon(\alpha_1+2) + \alpha_3(\varepsilon+1) - 1\right) + \alpha_2(1+\tau s) + \alpha_3 s\right],$$

$$a_2 = \frac{s}{\alpha_3 - 1}\left[(1+\tau_0 s^\alpha)\left(\varepsilon s\alpha_1^2 + \alpha_3 s + \alpha_2(\varepsilon+1)(1+\tau s)\right) + \alpha_2 s(1+\tau s)\right]$$

$$a_3 = \frac{s^4\alpha_2(1+s\ \tau)\ (1+s^\alpha\ \tau_0\ )}{\alpha_3 - 1}\ .$$

In a similar manner we can show that $\bar{e}$ and $\bar{C}$ satisfy the equations

$$(\mathscr{D}^6 - a_1\mathscr{D}^4 + a_2\mathscr{D}^2 - a_3)\bar{\theta} = 0\ ., \tag{31}$$

$$(\mathscr{D}^6 - a_1\mathscr{D}^4 + a_2\mathscr{D}^2 - a_3)\bar{C} = 0\ . \tag{32}$$

Equation (30) can be factorized as

$$(\mathscr{D}^2 - k_1^2)(\mathscr{D}^2 - k_2^2)(\mathscr{D}^2 - k_3^2)\bar{\theta} = 0\ , \tag{33}$$

where k_1, k_2 and k_3 are the roots with positive real parts of the characteristic equation

$$k^6 - a_1 k^4 + a_2 k^2 - a_3 = 0\ . \tag{34}$$

The solution of Eq. 33 has the form,

$$\bar{\theta}(x,s)=\sum_{i=1}^{3} A_i\, e^{-k_i x} \quad , \qquad (35)$$

where $A_i = A_i(s)$ are parameters depending on s only.Similarly, the solution of Eqs. 31 and 32 can be written as

$$\bar{e}(x,s)=\sum_{i=1}^{3} A_i'\, e^{-k_i x} \quad , \qquad (36)$$

$$\bar{C}(x,s)=\sum_{i=1}^{3} A_i''\, e^{-k_i x} \quad , \qquad (37)$$

where A_i'' are parameters depending only on s.

Substituting from equations (35)-(37) into equations (25), (26) and (29), we get

$$A_i' = \frac{k_i^2\,[k_i^2-(1-\varepsilon\alpha_2)(s+\tau_0 s^{1+\alpha})]}{\varepsilon(s+\tau_0 s^{1+\alpha})[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\, A_i \qquad (38)$$

$$A_i'' = \frac{k_i^4-k_i^2\,[s^2+(\varepsilon+1)(s+\tau_0 s^{1+\alpha})]+s^2(s+\tau_0 s^{1+\alpha})}{\varepsilon(s+\tau_0 s^{1+\alpha})[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\, A_i . \qquad (39)$$

We thus have

$$\bar{e}(x,s)=\sum_{i=1}^{3}\frac{k_i^2\,[k_i^2-(1-\varepsilon\alpha_2)(s+\tau_0 s^{1+\alpha})]}{\varepsilon(s+\tau_0 s^{1+\alpha})[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\left(A_i e^{-k_i x}+B_i e^{k_i x}\right) , \qquad (40)$$

$$\bar{C}(x,s)=\sum_{i=1}^{3}\frac{k_i^4-k_i^2\,[s^2+(\varepsilon+1)(s+\tau_0 s^{1+\alpha})]+s^2(s+\tau_0 s^{1+\alpha})}{\varepsilon(s+\tau_0 s^{1+\alpha})[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\times A_i e^{-k_i x} . \qquad (41)$$

Integrating both sides of equation (7) from x to infinity, and assuming that u vanishes at infinity, we obtain upon using the relation (40)

$$\bar{u}(x,s)=\sum_{i=1}^{3}\frac{k_i\,[k_i^2-(1-\varepsilon\alpha_1)(s+\tau_0 s^{1+\alpha})]}{\varepsilon(s+\tau_0 s^{1+\alpha})[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\left(-A_i e^{-k_i x}\right) . \qquad (42)$$

Substituting from equations (35), (40) and (41) into equations (27a) and (28), we get

$$\bar{\sigma}_{xx}(x,s)=\frac{s}{\varepsilon(1+\tau_0 s)}\sum_{i=1}^{3}\frac{[k_i^2-(1-\varepsilon\alpha_1)(s+\tau_0 s^{1+\alpha})]}{[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\left(A_i e^{-k_i x}\right) \qquad (43)$$

$$\bar{P}_{xx}(x,s)=\frac{\alpha_2(1+\tau s)}{\varepsilon(1+\tau_0 s^{\alpha})}\sum_{i=1}^{3}\frac{k_i^4-k_i^2\,[s^2+(\varepsilon+1)(s+\tau_0 s^{1+\alpha})]+s^2(s+\tau_0 s^{1+\alpha})}{[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\left(A_i e^{-k_i x}\right) . \qquad (44)$$

In order to evaluate the unknown parameters A_1, A_2 and A_3, we shall use the Laplace transform of the boundary conditions (21)-(23) together with equations (35), (43) and (44). We thus arrive at the following set of linear equations

$$\sum_{i=1}^{3}\frac{[k_i^2-(1-\varepsilon\alpha_1)(s+\tau_0 s^{1+\alpha})]}{[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\, A_i=0 , \qquad (45)$$

$$\sum_{i=1}^{3} A_i=\bar{f}_1(s) , \qquad (46)$$

$$\sum_{i=1}^{3}\frac{[k_i^4-k_i^2\,(s^2+(\varepsilon+1)(s+\tau_0 s^{1+\alpha}))+s^2(s+\tau_0 s^{1+\alpha})]}{k_i^2[(1+\alpha_1)k_i^2-\alpha_1 s^2]}\, A_i = \frac{\bar{f}_2(s)\,\varepsilon(1+\tau_0 s^{\alpha})}{\alpha_2(1+\tau s)} . \qquad (47)$$

Solving the linear system of equations (45)-(47), we can obtain the parameters A_1-A_3.This completes the solution of the problem in the Laplace transform domain.

INVERSION OF Laplace Transform

We shall now outline the numerical inversion method used to find the solution in the physical domain. Let $\bar{f}(x,s)$ be the Laplace transform of a function f(x, t). F

$$f(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}\,\overline{f}(x,s)ds$$

where c is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(s)$. Taking s = c + iy, the above integral takes the form

$$f(x,t)=\frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty}e^{ity}\,\bar{f}(x,c+iy)dy$$

Expanding the function h(x,y,t) = exp(-ct)f(x, t) in a Fourier series in the interval [0,2T], we obtain the approximate formula [23]

$$f(x,t)=f_{\infty}(x,t)+E_D,$$

where

$$f_{\infty}(x,t)=\tfrac{1}{2}c_0(x,t)+\sum_{k=1}^{\infty}c_k(x,t)\quad \text{for } 0\le t\le 2T, \tag{48}$$

and

$$c_k=\frac{e^{dt}}{T}\mathrm{Re}[e^{ik\pi t/T}\,\bar{f}(x,d+ik\pi/T)],\qquad k=0,1,2,.... \tag{49}$$

E_D, the discretization error, can be made arbitrarily small by choosing d large enough [15]. Since the infinite series in equation (48) can only be summed up to a finite number N of terms, the approximate value of f(x, t) becomes

$$f_N(x,t)=\tfrac{1}{2}c_0+\sum_{k=1}^{N}c_k\qquad \text{for } 0\le t\le 2T \tag{50}$$

Using the above formula to evaluate f(x, t), we introduce a truncation error E_T that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the “Korrecktur” method is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur method uses the following formula to evaluate the function f(x, t) :

$$f(x,t)=f_{\infty}(x,t)-e^{-2cT}f_{\infty}(x,2T+t)+E'_D,$$

where the discretization errors $|E'_D|<<|E_D|$ [23]. Thus, the approximate value of f(x, t) becomes

$$f_{NK}(x,t)=f_N(x,t)-e^{-2cT}f_{N'}(x,2T+t), \tag{51}$$

N´ is an integer such that N´ < N.

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in equation (53). Let N=2q+1 where q is a natural number, and let

$$s_m=\sum_{k=1}^{m}c_k$$

be the sequence of partial sums of equation (50). We define the ε-sequence by

$$\varepsilon_{0,m}=0,\ \varepsilon_{1,m}=s_m$$

and

εp+1,m = εp-1,m+1 + 1/ (εp,m+1-εp,m) , p=1,2,3,...

It can be shown that [23], the sequence

$\varepsilon_{1,1}$, $\varepsilon_{3,1}$, $\varepsilon_{5,1}$, ... , $\varepsilon_{N,1}$

converges to $f(x,t)+E_D-c_0/2$ faster than the sequence of partial sums

s_m , m = 1,2,3,...

The actual procedure used to invert the Laplace transforms consists of using equation (54) together

ISSN 2229-5518

with the ε-algorithm. The values of c and T are chosen according to the criteria outlined in [23].

3. NUMERICAL RESULTS

For the purpose of numerical illustration, the problem was solved for the following choice of the functions $f_1(t)$ and $f_2(t)$:

$$f_1(t) = \theta_0 H(t),$$

$$f_2(t) = P_0 H(t),$$

where θ_0 and P_0 are constants and $H(t)$ is the Heaviside unit step function.

We, thus, have

$$\bar{f}_1(s) = \frac{\theta_0}{s}$$

$$\bar{f}_2(s) = \frac{P_0}{s}$$

The roots k_1, k_2, and k_3 of the characteristic equation are given by

$$k_1 = \sqrt{\frac{1}{3}[2\, p \sin q + a_1]},$$

$$k_2 = \sqrt{\frac{1}{3}[a_1 - p(\sqrt{3}\cos q + \sin q)]},$$

$$k_3 = \sqrt{\frac{1}{3}[a_1 + p(\sqrt{3}\cos q + \sin q)]}$$

$$p = \sqrt{[(a_1^2 - 3a_2)]}, \quad q = \frac{\sin^{-1}(r)}{3}, \text{ and}$$

$$r = -\frac{2a_1^3 - 9a_1 a_2 + 27 a_3}{2p^3},$$

Copper material was chosen for purposes of numerical evaluations. The material constants of the problem are thus given by in SI units [24]

$T_0 = 293$ K, $\rho = 8{,}954$ kg.m^{-3}, $\tau_0 = 0.0\ 2$s, $\tau = 0\ .2$s, $c_E = 383.1$ J. kg $^{-1}$. K $^{-1}$, $\alpha_t = 1\ .7\ 8\times 10^{-5}$ K^{-1}, k = 386 W· m $^{-1}$· K^{-1}, $\lambda = 7\ .7\ 6\times 10^{10}$ kg · m $^{-1}$. s $^{-2}$, $\mu = 3.\ 8\ 6\times10^{10}$ kg · m $^{-1}$.s $^{-2}$, α c =1 .9 8× 10 $^{-4}$ m ^{3}. kg $^{-1}$, d =0.8 5× 10 $^{-8}$ kg.s.m^{-3}, a =1.2×10^4 m^2.s $^{-2}$.K $^{-1}$, b =0.9× 10^6 m 5· kg $^{-1}$. s $^{-2}$.

Using these values, it was found that

$\eta = 8886.73$, $\varepsilon = 0.0168$, $\beta^2 = 4$, $\alpha_1 = 5.43$, $\alpha_2 = 0.533$, and $\alpha_3 = 36.24$.

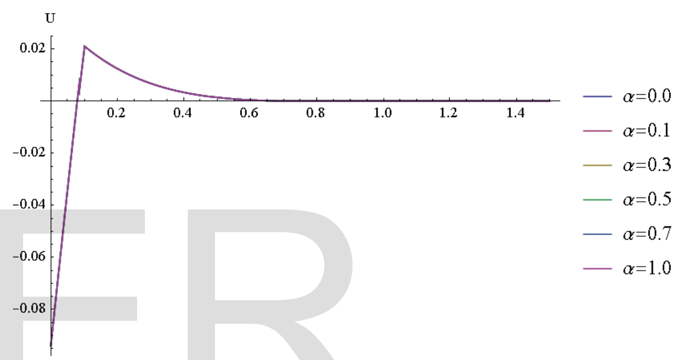

Figure 1 Displacement with all values of α

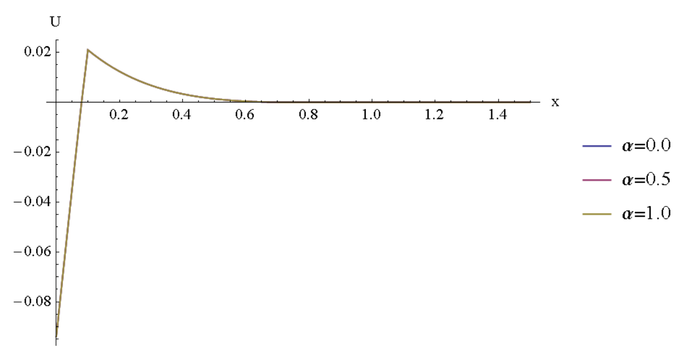

Figure 2 Displacement distribution for three values of α =0.0, α =0.5 and α =1.0.

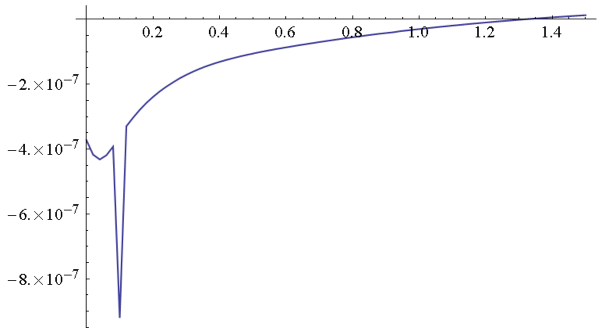

Figure 3 the difference between displacement distributions at values of α =0.0 and α =1.0

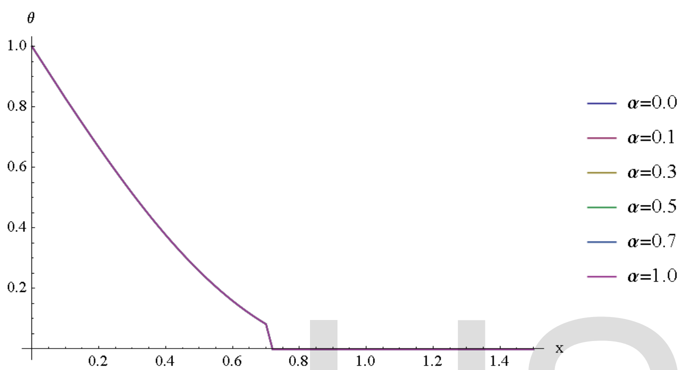

Figure 4 Temperature distribution with all values of α

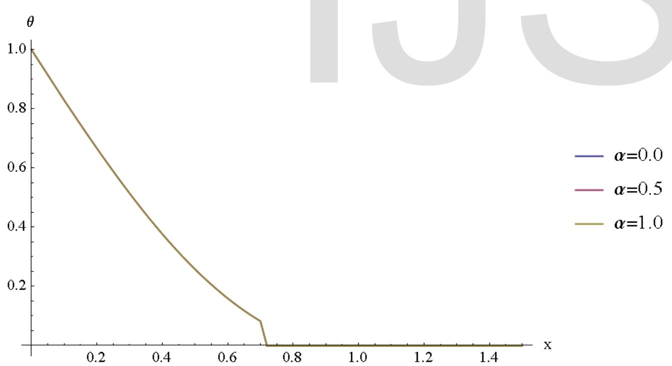

Figure 5 Temperature distribution for three values of α =0.0, α =0.5 and α =1.0.

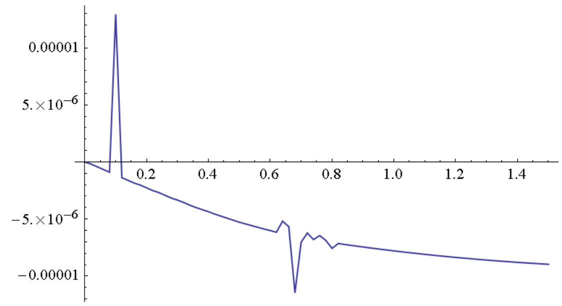

Figure 6 the difference between Temperature distribution at values of α =0.0 and α =1.0

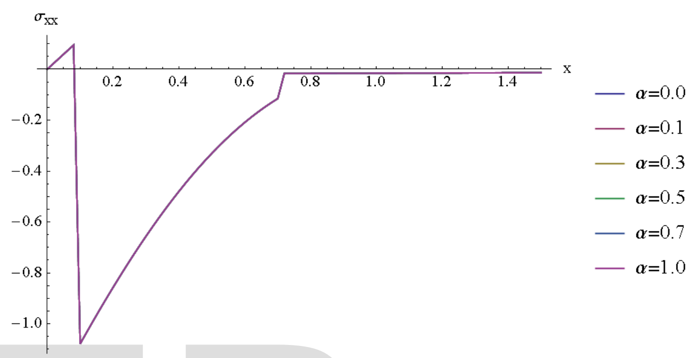

Figure 7 Stress Distribution with all values of α

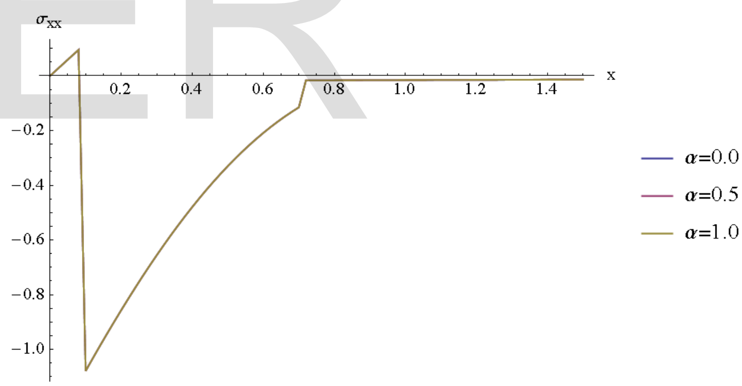

Figure 8 Stress Distribution for three values of α =0.0, α =0.5 and α =1.0.

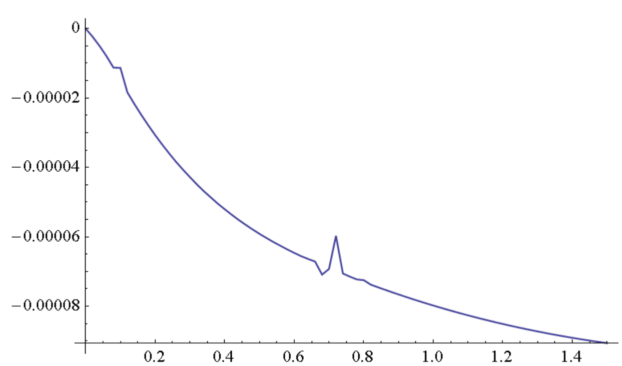

Figure 9 the difference between stress distribution at values of α =0.0 and α =1.0

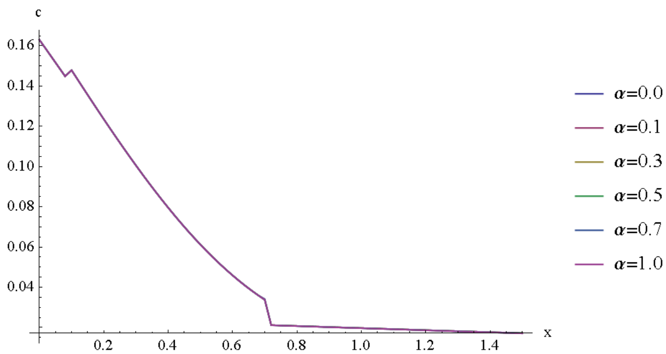

Figure 10 Concentration distribution all t values of α

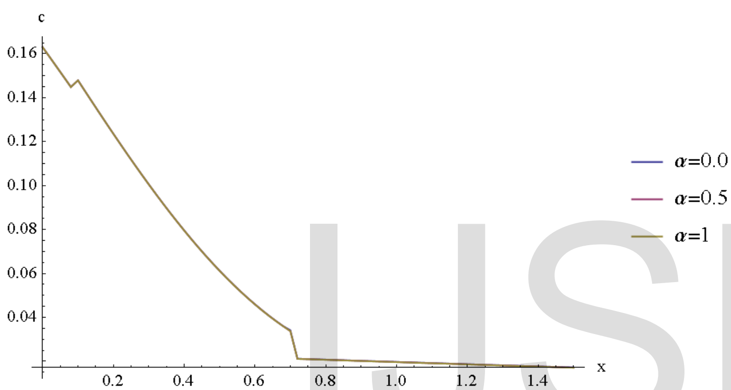

Figure 11 Concentration distribution at values of α =0.0 and α =1.0

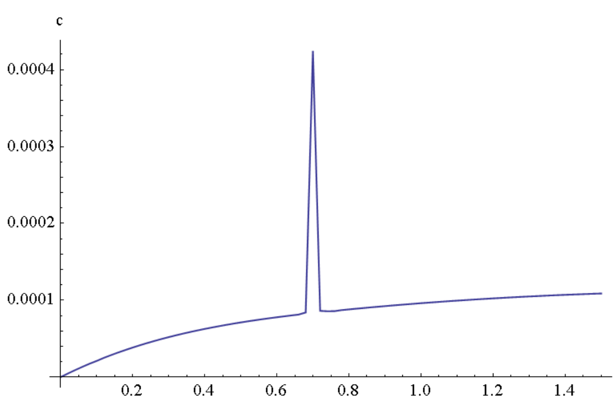

Figure 12 the difference between Concentration distribution at values of α =0.0 and α =1.0

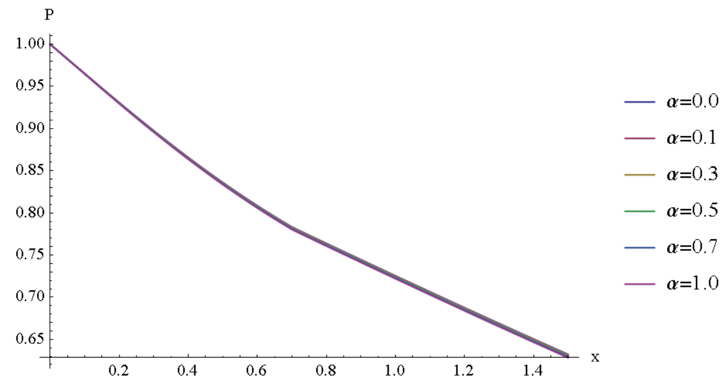

Figure 13 Chemical Potential distribution a all t values of α

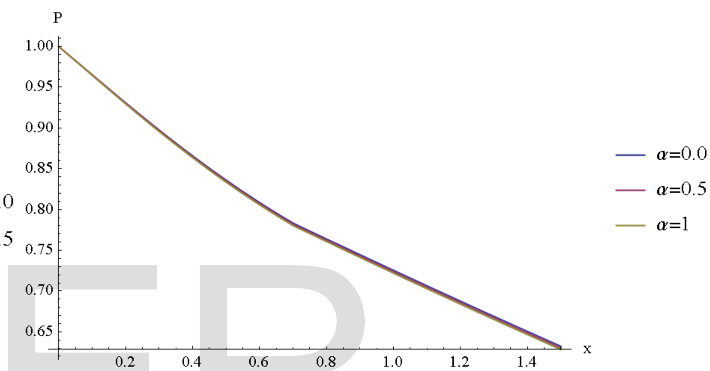

Figure 14 Chemical Potential distribution at values of α =0.0 and α =1.0

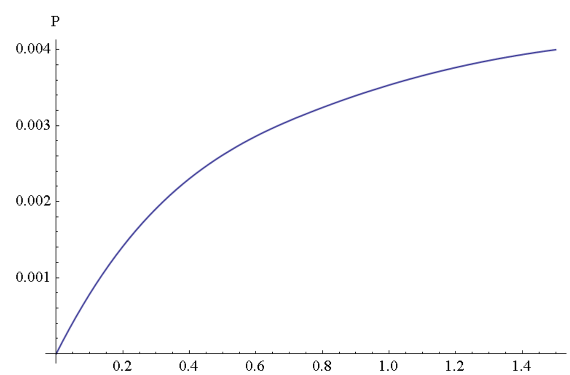

Figure 15 the difference between Chemical Potential distribution at values of α =0.0 and α =1.0

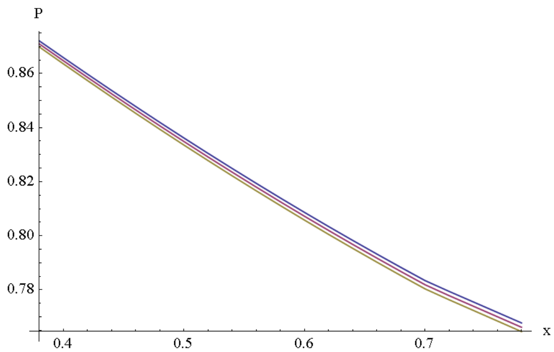

Figure 16 Chemical Potential distribution at values of α =0.0 and α =1.0

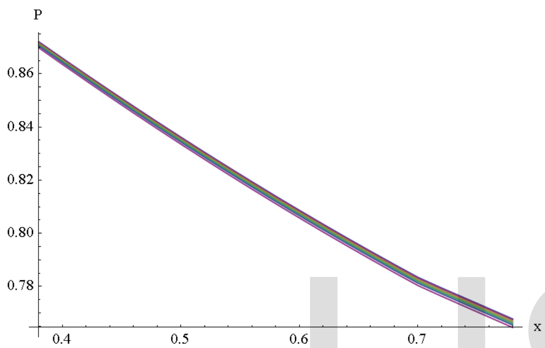

Figure 17 Chemical Potential distribution a all t values of α

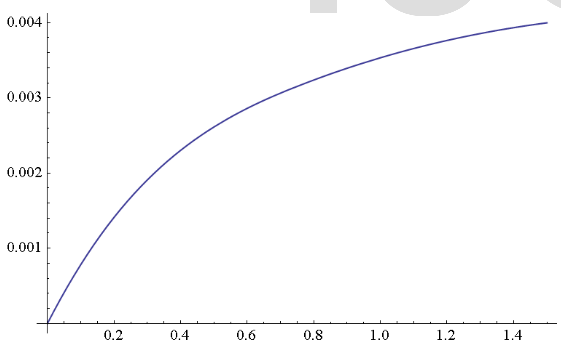

Figure 18 the difference between Chemical Potential at values of α =0.0 and α =1.0

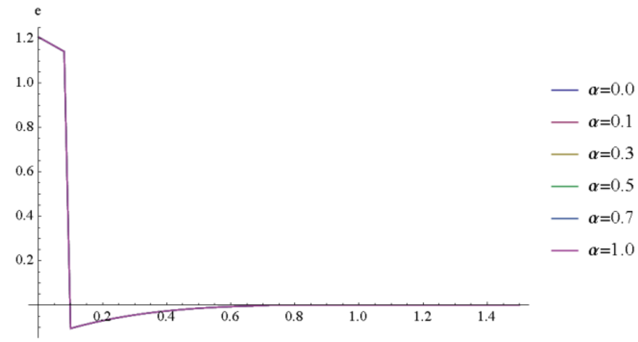

Figure 19 Strain distribution a all t values of α

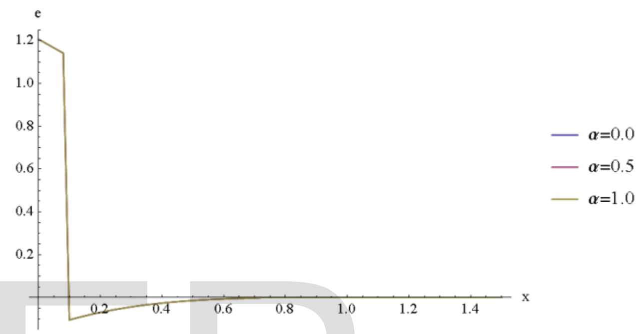

Figure 20 Strain distribution at values of α =0.0 and α =1.0

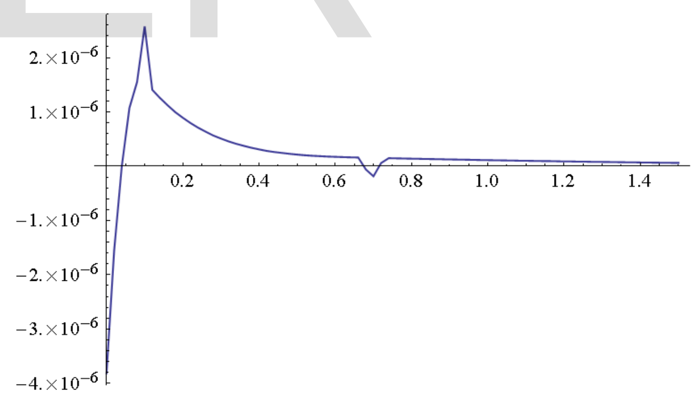

Figure 21 the difference between Strain distribution at values of α =0.0 and α =1.0

It should be noted that a unit of dimensionless time corresponds to 6.5 × 10−12 s, while a unit of dimensionless length corresponds to 2.7 × 10−8 m.

The computations were carried out for one value of time t = 0.1 and many values of α , from

0.0 to 1.0 especially two values of alpha $\alpha = 1$ (corresponding to LordShulman theory) and $\alpha = 0.5$. The numerical method outlined above was used to obtain the inverse Laplace transforms for the temperature, displacement and stress distributions. Fortran programming language was used on a personal computer. The accuracy maintained was 5 significant digits for both the numerical integration and the inversion of the Laplace transform.

The displacement, temperature, stress, concentration, chemical potential, and strain are shown in Figs. 1-3, 4-6, 7-9, 10-12, 13-18, and 19-21respectively.As expected from the order of the partial differential equation, we have three waves emanating from each surface; the fronts of these waves are depicted in the figures as picks in the functions .

We can see in all figures that, all the functions considered have a non-zero value only in a bounded region of space and vanish identically outside this region. This region expands with the passage of time. Also, our results are the sam as As was mentioned by Sherief in [14] when $\alpha = 1.0$, we here mentioned about the variation of α , from 0.0 to 1.0.

The displacement component u is shown in Fig. 1, for all values of α while fig. 2, show the difference between displacements at α=0.0 and α=1.0. It is clear that difference maintained was 7 significant digits, i.e the effect values of α are very weak, and we can see also in Fig.3 when, we choose three values of α($\alpha = 0.0$, $\alpha = 0.5$and $\alpha = 1$), the same coclusions are in temperature is shown in Figs. 4 to 6 , the stress component σxx in Figs. 7 to 9, chemical Concentration Figs. 10 to 12, and strain Figs. 19 to 21. But we note that in chemical potential Figs. 13 to 18 the effect for values of α maintained was 4 significant digits only .

We can say that, for $\alpha = 1$ the solution is that of the generalized theory of thermoelasticity and exhibits the phenomenon of finite speeds of propagation of waves. The question of whether the solution for $\alpha < 1$ behaves similarly or not is still an open question.

As was mentioned by Povstenko [17] ‘‘From numerical calculations, it is difficult to say whether the solution for a approaching 1 has a jump at the wave front or it is continuous with very fast changes. This aspect invites further investigation’’. Our calculations show that up to the specified accuracy, the solution seems to be non-zero only in a finite region of space.

The problem was solved for many values of $\alpha < 1$. The solutions seem to be of the same shape as that of $\alpha = 0.5$ reported here. The difference is that as a decreases the region where the solution is non-zero becomes larger indicating faster speed of propagation. It is known that for $\alpha = 0$, the solution is that of the coupled theory of thermoelasticity where the speed of propagation of the waves is infinite.

4-Acknowledgment:

The Authors present their deeply thanks for taif university for supporting us while doing in this research.

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